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Hamiltonian formulation of odd Burgers hierarchy

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Abstract. The derivation of the odd Burgers hierarchy is revisited. A first Hamiltonian formulation of the basic equation of the hierarchy is presented in relation with its linear counterpart. The generalized Poisson bracket is given explicitly. It contains exponentials of integrals of the dynamic variable. It verifies Jacobi identity by construction and through direct calculations. A second Hamiltonian formulation is also presented. It means that the equation, as expected, is 'bi-Hamiltonian'. This property permits, as usual, the construction of all the hierarchy. Extension to matrix Burgers systems is suggested.

1. Introduction

Odd members of the Burgers hierarchy appeared first in an IPP report of the author [1]. They are obtained by extending the 'linearization' achieved through the Cole–Hopf ansatz to equations containing as highest derivatives odd space derivatives. The most prominent example is given by

$$u_t = (u^3)_x + \tfrac{3}{2}(u^2)_{xx} + u_{xxx}. \tag{1}$$

In a subsequent report [2], the properties of the wave envelope solutions of this equation are investigated. Equation (1) also appeared in [3]. In parts of the literature (see for example [4–6]), it is called the Sharma–Tasso–Olver (STO) equation. In section 2, we show how equation (1) or STO can be linearized by the Cole–Hopf ansatz, and, in section 3, how this same ansatz leads to a Hamiltonian formulation of the equation. Section 4 contains a direct verification of the Jacobi identity for the Poisson bracket obtained in section 3. Section 5 deals with the second Hamiltonian formulation. Section 6 is devoted to the study of the constants of motion and to a discussion of the implications of the 'bi-Hamiltonian' property and possible generalizations.

2. Derivation of equation (1)

The Cole–Hopf ansatz relates the solutions of a nonlinear equation in u to the solutions of a specific linear equation in ϕ in the following way

$$u = \frac{\phi_x}{\phi} \tag{2}$$

or its inverse (see, for example, [7])

$$\phi = \exp D^{-1}u = \exp \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right) u(x')\,\mathrm{d}x' \tag{3}$$

where the definition of $D^{-1}u$ is given by the argument of the exponential in the second equality. In general, the time derivative of ϕ can be written as

$$\phi_t = A(u, u_x, \ldots)\phi. \tag{4}$$

The compatibility between equations (2) and (4) is warranted by

$$u_t = \frac{\partial A(u)}{\partial x} \tag{5}$$

if $A(u)$ is a function, and not an operator.

In fact, a sequence of functions $A_n(u)$ can be constructed by starting with

$$A_1 = u \tag{6}$$

and using the induction formula (see [8, 1])

$$A_{n+1} = A_n u + \frac{\partial A_n}{\partial x}. \tag{7}$$

In this way, we obtain for each value of n a nonlinear wave equation and its linear counterpart. For $n = 1$, we have

$$A_2 = u^2 + u_x \tag{8}$$

which leads to

$$u_t = (u^2)_x + u_{xx} \tag{9}$$

and its linear counterpart

$$\phi_t = \phi_{xx}. \tag{10}$$

Equation (8) is Burgers equation. Equation (10), which is the heat equation, is obtained after applying relation (2) to $A(u)\phi$ in (4).

For $n = 2$, we have

$$A_3 = (u^2 + u_x)u + (u^2 + u_x)_x \tag{11}$$

from which equation (1) follows as well as its linear counterpart

$$\phi_t = \phi_{xxx}. \tag{12}$$

For another derivation of equation (1), let us replace ϕ by its expression in u from relation (3). For that purpose, the derivatives of ϕ with respect to x and t are needed:

$$\phi_t = D^{-1}u_t \exp D^{-1}u \tag{13}$$

$$\phi_x = u \exp D^{-1}u \qquad \phi_{xx} = (u^2 + u_x) \exp D^{-1}u \tag{14}$$

$$\phi_{xxx} = [(u^2 + u_x)u + (u^2 + u_x)_x] \exp D^{-1}u. \tag{15}$$

After inserting relations (13) and (15) into equation (12), we take the derivative with respect to x, and recover equation (1). Similarly, other equations of the hierarchy can be constructed, and, in particular, the equations of the odd hierarchy. Note that mixed equations can also be constructed by taking $A = \sum_{n=1}^{N} c_n A_n$. For details, see [1].

3. First Hamiltonian formulation

For convenience, we are going to assume throughout the paper periodic boundary conditions for ϕ and u with $\int u\,\mathrm{d}x = 0$ on a period. The variations δu and $\delta\phi$ are taken equal to zero at the boundaries, as usual. Under these circumstances, equation (12), which is the linear counterpart of equation (1), is easy to cast in Hamiltonian form as follows

$$\phi_t = \frac{\partial}{\partial x}\frac{\delta H}{\delta \phi} \tag{16}$$

with

$$H = -\tfrac{1}{2}\int \phi_x^2\,\mathrm{d}x. \tag{17}$$

To find a Hamiltonian formulation of equation (1), we transform the Hamiltonian (17) according to (3) to obtain

$$H(u) = -\tfrac{1}{2}\int u^2(\exp 2D^{-1}u)\,\mathrm{d}x. \tag{18}$$

The next step is to express $\frac{\delta H}{\delta\phi}$ in terms of $\frac{\delta H}{\delta u}$. Let us calculate first the variation of $H(u)$ with respect to u

$$\delta H = -\int \mathrm{d}x\, u \exp(2D^{-1}u)\delta u + \int \mathrm{d}x\, [D^{-1}(u^2\exp(2D^{-1}u(x')))]\delta u. \tag{19}$$

Now, calculate the variation of ϕ with respect to u out of relation (3)

$$\delta\phi = (\exp D^{-1}u)D^{-1}\delta u. \tag{20}$$

From equation (20) it is easy to extract

$$\delta u = \frac{\partial}{\partial x}(\exp -D^{-1}u\delta\phi) \tag{21}$$

and insert it into equation (19), which becomes

$$\delta H = \int \mathrm{d}x\,\frac{\partial}{\partial x}[u\exp 2D^{-1}u]\exp(-D^{-1}u)\delta\phi - \int \mathrm{d}x\, u^2\exp D^{-1}u\delta\phi. \tag{22}$$

From equation (22) one can easily obtain

$$\frac{\delta H}{\delta\phi} = \left[(\exp -D^{-1}u)\frac{\partial}{\partial x}(u\exp 2D^{-1}u) - u^2\exp D^{-1}u\right]. \tag{23}$$

Taking equations (19) and (21) into account, equation (23) can be written as

$$\frac{\delta H}{\delta\phi} = -\exp -D^{-1}u\frac{\partial}{\partial x}\frac{\delta H}{\delta u}. \tag{24}$$

Finally, using equations (13) and (24) in (16), we obtain after taking the x derivative

$$u_t = -\frac{\partial}{\partial x}\exp -D^{-1}u\frac{\partial}{\partial x}\exp -D^{-1}u\frac{\partial}{\partial x}\frac{\delta H}{\delta u} \tag{25}$$

with the H given by equation (18).

Equation (25) can be more compactly written, if we introduce the Hamiltonian operator J

$$J = -D(\exp -D^{-1}u)D(\exp -D^{-1}u)D \tag{26}$$

where D is obviously the derivative with respect to x, and the 'generalized Poisson bracket'

$$\{F, G\} = \int \frac{\delta F}{\delta u}J\frac{\delta G}{\delta u}\,\mathrm{d}x \tag{27}$$

where F and G are any functionals of u. For more details about those definitions, see, for example, [9]. Definitions (26) and (27) allow equation (25) to be written in the form

$$u_t = J\frac{\delta H}{\delta u} = \{u, H\}. \tag{28}$$

Equation (28) together with the definitions (26) and (27) constitute a Hamiltonian formulation of equation (1). Note that bracket (27) should fulfil Jacobi identity by construction, i.e. due to the transformation (3). It is, however, instructive and safe to see by direct calculations whether the Jacobi identity is verified, which is the topic of the next section.

4. Verification of Jacobi identity

Bracket (27) is a generalized Poisson bracket (GPB) [9] if it fulfils the requirements of a Lie algebra. The first requirement is antisymmetry i.e.

$$\{F, G\} = -\{G, F\} \tag{29}$$

which can be seen by integrating by parts in (27).

The most important requirement is the fulfilment of Jacobi identity

$$\{E, \{F, G\}\} + \{G, \{E, F\}\} + \{F, \{G, E\}\} = 0. \tag{30}$$

If J were independent upon u, the antisymmetry of GPB (27) would suffice to verify Jacobi identity (30). In fact, the functional derivative of $\{F, G\}$ contains three terms

$$\frac{\delta\{F,G\}}{\delta u} = \int \mathrm{d}x \left(\frac{\delta F}{\delta u}\right)'_u J\frac{\delta G}{\delta u} + \int \mathrm{d}x\, \frac{\delta F}{\delta u} J \left(\frac{\delta G}{\delta u}\right)'_u + \int \mathrm{d}x\, \frac{\delta F}{\delta u} J'_u \frac{\delta G}{\delta u}. \tag{31}$$

The two first terms of (31) contain second-functional derivatives. This and the antisymmetry of J cause their cancellation with the other corresponding terms in the Jacobi identity (30), as mentioned in [9]. The last term contains the Fréchet derivative of the operator J. The cancellation of this term in (30) is not obvious, so that its calculation is required (see [9]). If we call this rest term R, we find

$$R = \int \mathrm{d}x\, \frac{\delta E}{\delta u}\frac{\partial}{\partial x}(\exp -D^{-1}u)\frac{\partial}{\partial x}(\exp -3D^{-1}u)\left[\left(\frac{\delta F}{\delta u}\right)_{xx}\left(\frac{\delta G}{\delta u}\right)_x - \left(\frac{\delta F}{\delta u}\right)_x\left(\frac{\delta G}{\delta u}\right)_{xx}\right]$$
$$+(\text{cyclic permutations of } E,\ F \text{ and } G). \tag{32}$$

Let us integrate equation (32) twice by parts to obtain

$$R = \int \mathrm{d}x\, (\exp -4D^{-1}u)\left[\left(\frac{\delta E}{\delta u}\right)_{xx} - u\left(\frac{\delta E}{\delta u}\right)_x\right]\left[\left(\frac{\delta F}{\delta u}\right)_{xx}\left(\frac{\delta G}{\delta u}\right)_x - \left(\frac{\delta F}{\delta u}\right)_x\left(\frac{\delta G}{\delta u}\right)_{xx}\right]$$
$$+(\text{cyclic permutations of } E,\ F \text{ and } G). \tag{33}$$

It is straightforward to see that all cyclic permutations in equation (33) cancel each other, so that $R = 0$. Jacobi identity is, indeed, verified.

5. Second Hamiltonian formulation

Since equation (1) and its linear counterpart (12) are completely integrable, a second Hamiltonian formulation in the same dynamic variable, but with a different Hamiltonian operator and a different Hamiltonian, should be possible [10]. Indeed, equation (12) can be written as

$$\phi_t = \frac{\partial^3}{\partial x^3}\frac{\delta H_1}{\delta \phi} \tag{34}$$

with

$$H_1 = \frac{1}{2}\int \mathrm{d}x\, \phi^2. \tag{35}$$

Similarly to section 3, we express $\frac{\delta H_1}{\delta\phi}$ in terms of $\frac{\delta H_1}{\delta u}$ to obtain

$$\frac{\delta H_1}{\delta \phi} = -\exp(-D^{-1}u)\frac{\partial}{\partial x}\frac{\delta H_1}{\delta u}. \tag{36}$$

Now, using equations (13) and (36) in (34), we obtain after taking the x derivative

$$u_t = -\frac{\partial}{\partial x}\exp(-D^{-1}u)\frac{\partial^3}{\partial x^3}\exp(-D^{-1}u)\frac{\partial}{\partial x}\frac{\delta H_1}{\delta u} \tag{37}$$

where

$$H_1 = \frac{1}{2}\int \mathrm{d}x\, \exp 2D^{-1}u. \tag{38}$$

The new Hamiltonian operator is given by

$$K = -D\exp(-D^{-1}u)D^3\exp(-D^{-1}u)D \tag{39}$$

and its associated GPB is

$$\{F, G\} = \int \frac{\delta F}{\delta u} K \frac{\delta G}{\delta u}\,\mathrm{d}x. \tag{40}$$

The verification of Jacobi identity for K is accomplished in a way similar to the case with J of section 4.

6. Constants of motion and discussion

Equation (12), which is the linear counterpart of equation (1) has an infinity of constants of motion:

$$\int \mathrm{d}x\,\phi \qquad \int \mathrm{d}x\,\phi^2 \qquad \int \mathrm{d}x\,\phi_x^2 \qquad \int \mathrm{d}x\,\phi_{xx}^2 \qquad \text{etc.} \tag{41}$$

Equation (12) is, in some sense, completely integrable because the constants of motion are in involution i.e. all their mutual GPB vanish. Through equations (3), (14) and (15), it is easy to express constants of motion (41) in terms of u. It is also easy to check that they are in involution, using GPB (27). Note that $C = \int \mathrm{d}x\, u$ would have been the only Casimir of bracket (27), if it were not identically zero due to the choice of space of functions of section 3.

The derivation of the Hamiltonian formulation given in section 3 concerns equation (1). It is, however, easy to see that a similar derivation can be carried through for the higher members of the odd hierarchy of Burgers equation. An interesting feature is that the GPB (27) remains unchanged through the hierarchy. What obviously changes is the Hamiltonian.

These remarks apply also to the second Hamiltonian formulation given in section 5. In fact, the whole hierarchy can be mapped, in principle [11], by the recurrence operator $R = KJ^{-1}$, since the system is bi-Hamiltonian. Finally, it would be interesting to know whether these Hamiltonian formulations can be extended to the conservative part of the 'matrix Burgers' hierarchy introduced in [12].

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